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On Centrally Regular Modules and Centrally Semiregular Modules.

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Abstract

In this paper, two new modules are defined, which we call centrally regular and centrally semiregular modules and several properties of them are proved. Also, we have determined so many conditions under which regular (resp. semiregular) modules and centrally regular (resp. centrally semiregular) modules are equivalent.

Keywords: regular modules, semiregular modules, small modules,centrally regular modules, centrally semiregular modules.

Introduction

Let R be a ring with identity 1 and M be a left $R-$module. A nonempty subset S of R is called a multiplicative system in R if $0 \notin S$ and $a,b \in S$ implies $ab \in S$ [1]. If S is a multiplicative system in R such that $[S,R]=\{0\}$,where $[S,R]=\{[s,r]: s \in S, r \in R\}$ and $[s,r]=sr-rs$, then one can easily show that $R_S=\{a_m : a \in R, m \in S\}$ is a ring under the following operations of addition and multiplication: **(i):** $a_m + b_n = (na+mb)_{mn}$ and **(ii):** $a_m b_n = (ab)_{mn}$, for all $a_m, b_n \in R_S$ [2] and this ring is known as the ring of quotients of R with respect to the multiplicative system S or the localization of R at the multiplicative system S, where a_m is the equivalence of (a,m) in $R \times S$ under the equivalence relation (~) defined as follows: If $(a,m),(b,t) \in R \times S$ then $(a,m) \sim (b,t)$ if and only if there exists $s \in S$ such that $s(ta-mb)=0$ and also we would like to mention that m_m is the identity element of R_S for all $m \in S$ [2]. Also it can be shown that $M_S=\{m_s : m \in M, s \in S\}$ is a left R_S- module under the module operations defined as $a_m + b_t = (ta+mb)_{mt}$ and $\lambda_p a_m = (\lambda a)_{pm}$, for all $\lambda_p \in R_S$ and $a_m, b_t \in M_S$, where m_s is the equivalence of (m,s) in $M \times S$ under the equivalence relation (~) which is defined as follows: If $(a,m),(b,t) \in M \times S$ then $(a,m) \sim (b,t)$ if and only if there exists $s \in S$ such that $s(ta-mb)=0$.

A submodule K of M is called a cyclic submodule of M if $K=Rx$, for some $x \in M$ [3] and it is called a small submodule of M if $K+L \neq M$, for every proper submodule L of M, or equivalently, if L is any submodule of M such that $K+L=M$, then $L=M$ [4-6]. Also, we say that K lies above a direct summand of M if there is a direct decomposition $M=P \oplus Q$ such that $P \subseteq K$ and $Q \cap K$ is a small submodule

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of Q [7,8]. M is called a regular module if every cyclic submodule of M is a direct summand of M [9] and it is called a semiregular module if every cyclic submodule of M lies above a direct summand of M [9]. Also M is said to be simple if the only submodules of M are $\{0\}$ and M [10]. The Jacobson radical of M, denoted by $J(M)$, is the intersection of all the maximal submodules of M [3] and M is called semiprimitive if $J(M)=0$ [9].

Remarks

If S is a multiplicative system in R with $[S,R]=\{0\}$, then:

1: For all $s \in S$, we have 0_s is the zero of M_S, where 0 is the zero of M and $0_m = 0_n$, for all $m,n \in S$. Also, for all $s \in S$, we have s_s is the identity element of R_S and it is easy to see that $m_m = n_n$ for all $m,n \in S$ [2].

2: If $a_m, b_t \in R_S$, where $a,b \in R$ and $m,t \in S$, then $a_m = b_t$ if and only if $(a,m) \sim (b,t)$ if and only if there exists $s \in S$ such that $s(ta-mb)=0$ and $a_m = 0$ if and only if there exists $u \in S$ such that $ua=0$ [2]. The same result is valid if we replace R by M and R_S by M_S.

3: If $r,s \in R$ and $m,n \in S$, then we have $s_n + (-s)_n = (ns-ns)_{nn} = 0_{nn} = 0_n$ and so

$(-s)_n = -s_n$ .Now $(r+s)_m = m_m(r+s)_m =$ $(mr+ms)_{mm} = r_m + s_m$ and thus
$(r-s)_m = (r+(-s))_m =$
$r_m + (-s)_m = r_m - s_m$ [2].

4: If $a,b \in M$ and $m,n \in S$, then we have $b_n + (-b)_n = (nb+n(-b))_{nn} = 0_{nn} = 0_n$ and so . Also, $(a+b)_m = m_m(a+b)_m =$ $(ma+mb)_{mm} = a_m + b_m$ and
$(a-b)_m = (a+(-b))_m =$
$a_m + (-b)_m = a_m - b_m$.

Known Results

Theorem A: [9]

Let R be a ring with identity and M an $R-$module, then the following conditions are equivalent:

1: M is regular.

2: Every finitely generated submodule of M is a direct summand of M.

3: For every finitely generated submodule N of M there exists an idempotent endomorphism f of M such that $f(M)=N$.

Theorem B: [9]

Let M be an $R-$module, then the following conditions are equivalent:

1: M is semiregular.

2: Every finitely generated submodule of M lies above a direct summand of M.

3: For every cyclic submodule N of M, there is an idempotent endomorphism f of M such that $f(M) \subseteq N$ and $(1-f)(N)$ is a small submodule of M.

4: For every finitely generated submodule N of M there exists an idempotent endomorphism f of M such that such that $f(M) \subseteq N$ and $(1-f)(N)$ is a small submodule of M.

Theorem C: [9]

Let M be an $R-$module. If $J(M)$ is a small submodule of M, then the following conditions are equivalent:

1: M is semiregular.

2: For every cyclic submodule N of M, there is an idempotent endomorphism f of M such that $f(M) \subseteq N$ and

$(1-f)(N) \subseteq J(M)$.

3: For every finitely generated submodule N of M there exists an idempotent endomorphism f of M such that $f(M) \subseteq N$ and $(1-f)(N) \subseteq J(M)$.

Theorem D: [9]

Let M be an $R-$module, then the following conditions are equivalent:

1: M is semiprimitive semiregular module.

2: M is a regular module.

3: Every submodule of M is a semiprimitive regular module.

Theorem E: [3]

An $R-$module M is simple if and only if $M = Rx$, for all $0 \neq x \in M$.

Theorem F: [9]

A submodule K of M is a maximal submodule in M if and only if $\frac{M}{K}$ is a simple $R-$module.

Now we introduce the following definition.

Definition:

We call a multiplicative system S in R a central multiplicative system if $[S,R]=\{0\}$.

It is necessary to mention that $[S,R]=\{0\}$ if and only if $S \subseteq Z(R)$, where $Z(R)$, is the center of the ring R, that is, $sr = rs$, for all $s \in S, r \in R$.

The Main Results:

First, we give the following lemma. Its proof is simple and follows directly by the elementary properties of localization and it is basic in driving our main results.

Lemma 1:

Let K and N be submodules of M and S a central multiplicative system in R.

(1): If $K \subseteq N$, then $K_S \subseteq N_S$ and hence, if $K = N$, then $K_S = N_S$.

(2): $(K+N)_S = K_S + N_S$.

(3): If L is any other submodule of M such that $K + N = L$, then $K_S + N_S = L_S$.

(4): $(K \cap L)_S = K_S \cap L_S$.

Next, we prove several results, which concerning the submodules of M and the submodules of the localized module M_S and which will lead to the proof of our first main theorem.

Lemma 2:

Let S be a central multiplicative system in R. If $a \in M$ and $m \in S$, then $R_S a_m = (Ra)_S$.

Proof:

Let $x' \in R_S a_m$. Then there exists $r_t \in R_S$, such that $x' = r_t a_m = (ra)_{tm} \in (Ra)_S$ (because $ra \in Ra$ and $tm \in S$).Hence $R_S a_m \subseteq (Ra)_S$. Let $y' \in (Ra)_S$, then there exists $s \in R$ and $p \in S$ such that $y' = (sa)_p$. Then we get $y' = (sa)_p = (sa)_p m_m = (sm)_p a_m \in R_S a_m$ (since $(sm)_p \in R_S$ and $R_S a_m$ is a submodule of M_S and hence itself is an R_S- module). So $(Ra)_S \subseteq R_S a_m$. Hence $R_S a_m = (Ra)_S$ □.

Proposition 3:

Let S be a central multiplicative system in R. If K is a submodule of M, then K_S is a submodule of M_S.

Proof:

$K_S = \{a_m : a \in K, m \in S\}$.Take $s \in S$ (since $S \neq \phi$) and since $0 \in K$, so

$0_S \in K_S$, that is, K_S contains the zero of M_S. Now if $a_m, b_n \in K_S$ and $\lambda_p \in R_S$, where $a, b \in K$, $\lambda \in R$ and $m, n, p \in S$, then we have $a_m - b_n = (na - mb)_{mn} \in K_S$ and $\lambda_p a_m = (\lambda a)_{pm} \in M_S$ (since $na - mb, \lambda a \in K$). So K_S is a submodule of M_S □.

Proposition 4

Let S be a central multiplicative system in R and $s \in S$ be any element. If K^* is a submodule of M_S, then $K = \{a \in M : (sa)_s \in K^*\}$ is a submodule of M and $K^* = K_S$, that is, for each submodule K^* of M_S, there exists a submodule K of M such that $K^* = K_S$.

Proof

Since $0 \in M$ and $(s0)_s = 0_s \in K^*$, so $0 \in K$. Hence $\phi \neq K \subseteq M$. Now let $a, b \in K$ and $\lambda \in R$. Then $(sa)_s, (sb)_s \in K^*$. Also we have $(s(a-b))_s = (sa - sb)_s = (sa)_s - (sb)_s \in K^*$. Hence $a - b \in K$.

Next we have $(s(\lambda a))_s = (s(\lambda a))_s s_s = (\lambda s)_s (sa)_s \in K^*$, so $\lambda a \in K$. Hence K is a submodule of M. It remains to show that $K^* = K_S$. Let $a_m \in K^*$, where $a \in M$ and $m \in S$. Then we have $(sa)_s = m_m (sa)_s = (ms)_s a_m \in K^*$ and hence $a \in K$. So that $a_m \in K_S$ and thus $K^* \subseteq K_S$. Let $a_m \in K_S$, where $a \in K$ and $m \in S$. Then $(sa)_s \in K^*$. Now $a_m = s_s a_m = s_s (1a)_m = 1_m (sa)_s \in K^*$ (since $1_m \in R_S$ and $(sa)_s \in K^*$ and K^* is an $R_S -$ module). Hence $K_S \subseteq K^*$ and thus $K^* = K_S$ □.

Remark

It is necessary to mention that, in above proposition the choice of s does not depend on the submodule K, since for all $s, t \in S$, we have $(sa)_s = t_t (sa)_s = s_s (ta)_t = (ta)_t$. More general, for any $s, t \in S$, we have $\{a \in M : (sa)_s \in K^*\} = \{b \in M : (tb)_t \in K^*\}$. To prove this, let $x \in \{a \in M : (sa)_s \in K^*\}$, then $(sx)_s \in K^*$. We get $(tx)_t = (sx)_s \in K^*$ and thus $x \in \{a \in M : (ta)_t \in K^*\}$, so that $\{a \in M : (sa)_s \in K^*\} \subseteq \{b \in M : (tb)_t \in K^*\}$. Similarly we can show $\{b \in M : (tb)_t \in K^*\} \subseteq \{a \in M : (sa)_s \in K^*\}$. Hence $\{a \in M : (sa)_s \in K^*\} = \{b \in M : (tb)_t \in K^*\}$. As especial case, if $1 \in S$, then by taking $s = 1 \in S$, the submodule K of **Proposition 4**, becomes $K = \{c \in M : c_1 \in K^*\}$. Hence in the case when $1 \in S$, we get $\{a \in M : (sa)_s \in K^*\} = \{b \in M : (tb)_t \in K^*\} = \{c \in M : c_1 \in K^*\}$.

Lemma 5

Let K and L be submodules of M and every non zero element of $Z(R)$ is a unit in R.

1: If $x \in K, y \in L$ and $sx = ty$, for some

$s, t \in S$, then $x \in L$ and $y \in K$.

2: If $K_S = L_S$, then $K = L$. As especial case, if $K_S = 0$, then we have $K = 0$.

Proof

1: The proof will follows directly by the fact that $S \subseteq Z(R)$ and thus s and t are non zero elements of $Z(R)$, so they are units and hence $x = s^{-1}ty \in L$ and $y = t^{-1}sx \in K$.

2: Since $K_S = L_S$, so $K_S \subseteq L_S$ and $L_S \subseteq K_S$. If $K_S \subseteq L_S$, to show $K \subseteq L$. If $x \in K$, then for a fixed $u \in S$, we have $x_u \in K_S \subseteq L_S$. Hence there exists $l \in L$ and $t \in S$ such that $x_u = l_t$ and thus there exists $s \in S$ such that $stx = sul$. Then from the result of (**1**), we get $x \in L$ and hence $K \subseteq L$. Similarly, it can be shown that $L \subseteq K$. Hence $K = L$. For the second part, if $K_S = 0 = 0_S$, then we get $K = 0$ □.

Now we introduce the following definition:

Definition

We say that, M is a centrally regular $R-$module if the localization M_S of M at every central multiplicative system S in R is a regular R_S-module.

Theorem 6

If every non zero element of $Z(R)$ is a unit in R, then M is regular if and only if it is centrally regular.

Proof

Suppose that M is regular and S is any central multiplicative system in R. If $R_S a_m$ is any cyclic submodule of M_S, for $a \in M$ and $m \in S$ then by **Proposition 4**, we get $R_S a_m = K_S$, for the submodule $K = \{a \in M : (sa)_S \in R_S a_m\}$ of M, where $s \in S$ is a fixed element in S. Now by **Lemma 2**, we have $R_S a_m = (Ra)_S$. Hence $K_S = (Ra)_S$. To show $K = Ra$. Let $x \in K$, then $(sx)_S \in K_S = (Ra)_S$. Hence there exists $r \in R$ and $n \in S$ such that $(sx)_S = (ra)_n$, which implies that there exists $b \in S$ such that $b(nsx - sra) = 0$. Then $bnsx = bsra$. Then, by **Lemma 5**, we get $x \in Ra$. Hence $K \subseteq Ra$. Now let $y \in Ra$, so $y = ta$, for some $t \in R$, then $(sy)_S = (sta)_S \in (Ra)_S = K_S = R_S a_m$. Thus $y \in K$, so that $Ra \subseteq K$. Hence $K = Ra$, that means K is a cyclic submodule of the regular left $R-$module M, so that there exists a submodule N of M such that $M = K \oplus N$. That is, $M = K + N$ and $K \cap N = \{0\}$. Then by **Lemma 1**, we get $M_S = K_S + N_S$. To show $K_S \cap N_S = \{0\}$. Let $x' \in K_S \cap N_S$, then there exists $k \in K, l \in N$ and $u, v \in S$ such that $x' = k_u = l_v$ and so there exists $w \in S$ such that $w(vk - ul) = 0$, that is, $wvk = wul \in K \cap N = \{0\}$. Hence $wvk = wul = 0$. Then we get $x' = k_u = w_w v_v k_u = (wvk)_{wvu} = 0_{wvu} = 0$, so that $K_S \cap N_S = \{0\}$ and thus $M_S = K_S \oplus N_S = R_S a_m \oplus N_S$. Hence $R_S a_m$ is a direct summand of M_S and thus M_S is a regular left R_S-module.

Conversely, let M be a centrally regular $R-$module. Consider $S = Z(R) - \{0\}$. It is easy to see that $S = Z(R) - \{0\} \subseteq Z(R)$ is a central multiplicative system in R and since M is centrally regular, so M_S is a regular left R_S – module. To show M is regular. Let Rx be any cyclic submodule of M, where $x \in M$. Since $1 \neq 0$ and $1 \in Z(R)$, so $1 \in S$. Using **Lemma 2**, we get $R_S x_1 = (Rx)_S$. Now $R_S x_1$ is a cyclic submodule of the regular left R_S – module M_S. Hence $M_S = R_S x_1 \oplus K^*$, for some submodule K^* of M_S. Then by **Proposition 4**, $K^* = K_S$ for the submodule $K = \{a \in M : a_1 \in K^*\}$ of M (see the last remark). Hence, the last result yields $M_S = (Rx)_S \oplus K_S$, that is $M_S = (Rx)_S + K_S$ and $(Rx)_S \cap K_S = \{0\}$. To show $M = Rx \oplus K$. Let $y \in M$. As $1 \in S$, we get $y_1 \in M_S$, so there exists $r \in R, k \in K$ and $u, v \in S$ such that $y_1 = (rx)_u + k_v = (vrx + uk)_{uv}$. Then there exists $w \in S$ such that $wuvy = wvrx + wuk$. Since $u, v, w \in S$ and $S = Z(R) - \{0\} \subseteq Z(R)$, so u, v, w all are non zero elements of the center $Z(R)$ and thus they are all units of R. Hence $u^{-1}, v^{-1}, w^{-1} \in R$ and so from the last result we get $y = u^{-1}rx + v^{-1}k \in Rx + K$, so that $M \subseteq Rx + K$. Hence $M = Rx + K$. Next to show that $Rx \cap K = \{0\}$. Let $b \in Rx \cap K$. So there exists $t \in R$ and $l \in K$ such that $b = tx = l$. Then we get $b_1 = (tx)_1 = l_1 \in (Rx)_S \cap K_S = \{0\}$, that is, $b_1 = 0$ and so there exists $c \in S$ such that $cb = 0$. Since c is a non zero element of $Z(R)$, so it is a unit and thus $c^{-1} \in R$. Hence we get $b = c^{-1}cb = c^{-1}0 = 0$. Therefore, $Rx \cap K = \{0\}$ and thus $M = Rx \oplus K$. Hence M is a regular R – module □.

Combining **Theorem A** with **Theorem 6**, we get the following theorem:

Theorem 7

If R is a division ring, then the following conditions are equivalent:

1: M is regular.
2: M is centrally regular.
3: Every finitely generated submodule of M is a direct summand of M.
4: For every finitely generated submodule N of M there exists an idempotent endomorphism f of M such that $f(M) = N$.

Proof:

Since every non zero element of a division ring is a unit and $Z(R) \subseteq R$, so every non zero element of $Z(R)$ is also a unit and hence the result will follows directly from **Theorem 6** □.

Now we prove the following proposition which leads to the proof of the next results of the paper.

Proposition 8

Let K be a submodule of M and S is any central multiplicative system in R. If every non zero element of $Z(R)$ is a unit in R, then:

1: K is a small submodule of M if and only if K_S is a small submodule of M_S.

2: $\frac{M}{K}$ is a simple left R – module if and only if $\frac{M_S}{K_S}$ is a simple left R_S – module and in either cases we have

$\frac{M_S}{K_S} = (\frac{M}{K})_S$.

3: K is a maximal submodule of M if and only if K_S is a maximal submodule of M_S.

4: M is a simple $R-$module if and only if M_S is a simple R_S-module.

5: $J(M_S) = (J(M))_S$.

6: $J(M) = 0$ if and only if $J(M_S) = 0$, that is, M is semiprimitive if and only if M_S is semiprimitive.

7: K is a cyclic submodule of M if and only if K_S is a cyclic submodule of M_S.

Proof

1: Let K be a small submodule of M and S is any central multiplicative system in R. To show K_S is a small submodule of M_S. Let L^* be any submodule of M_S such that $K_S + L^* = M_S$. Then by **Proposition 4**, we get $L^* = L_S$, for the submodule $L = \{a \in M : (sa)_S \in L^*\}$ of M, where s is a fixed element in S. Hence we get $K_S + L_S = M_S$ and from **Lemma 1**, we get $(K+L)_S = M_S$. Thus from **Lemma 5**, we get $K + L = M$. Since K is a small submodule of M, so $L = M$, and then by **Lemma 1**, we get $L_S = M_S$, that is, $L^* = M_S$. Hence K_S is a small submodule of M_S.

Conversely, suppose that K_S is a small submodule of M_S. To show K is a small submodule of M. Let L be any submodule of M such that $K + L = M$. It is easy to show that $S = Z(R) - \{0\}$ is a central multiplicative system in R. So by **Lemma 1**, we get $K_S + L_S = M_S$ and since K_S is small in M_S, so $L_S = M_S$ and hence by **Lemma 5**, we get $L = M$ and thus K is a small submodule of M.

2: Let $\frac{M}{K}$ be a simple $R-$module. To show $\frac{M_S}{K_S}$ is a simple R_S-module. Let $m_S + K_S$ be any non zero element of $\frac{M_S}{K_S}$, where $m \in M, s \in S$, that is, $m_S + K_S \neq K_S$. Hence $m_S \notin K_S$. Then $m \notin K$ (otherwise, $m_S \notin K_S$). Thus $m + K \neq K$, that means $m + K$ is a non zero element of the simple $R-$module $\frac{M}{K}$. Using **Theorem E**, we get $\frac{M}{K} = R(m+K) = Rm + K$. To show $\frac{M_S}{K_S} = (Rm)_S + K_S$. Let $a_t + K_S \in \frac{M_S}{K_S}$, for $a \in M, t \in S$. Then we get $a + K \in \frac{M}{K} = Rm + K$. Hence there exists $r \in R$ such that $a + K = rm + K$, so that $a - rm \in K$. Then $a_S - (rm)_S = (a - rm)_S \in K_S$. Hence $a_S + K_S = (rm)_S + K_S \in (Rm)_S + K_S$. So we have $\frac{M_S}{K_S} \subseteq (Rm)_S + K_S$. Since $Rm \subseteq M$, so $(Rm)_S \subseteq M_S$ and thus $(Rm)_S + K_S \subseteq \frac{M_S}{K_S}$. Hence $\frac{M_S}{K_S} = (Rm)_S + K_S$. From **Lemma 2**, we have $R_S m_s = (Rm)_S$, so that we get $\frac{M_S}{K_S} = R_S m_s + K_S = R_S(m_s + K_S)$. Hence $\frac{M_S}{K_S}$ is a simple $R_S -$module. By using **Lemma 1**, we get $\frac{M_S}{K_S} = (Rm)_S + K_S = (Rm + K)_S =$ $(\frac{M}{K})_S$. Now let $\frac{M_S}{K_S}$ be a simple $R_S -$module. To show $\frac{M}{K}$ is a simple $R -$module. Let $x + K$ be a non zero element of $\frac{M}{K}$, that is, $x + K \neq K$. Hence $x \notin K$. As $S \neq \phi$, fix an $u \in S$. If $x_u \in K_S$, then there exists $k \in K, t \in S$ such that $x_u = k_t$. So there exists $v \in S$ such that $vtx = vuk$. Then by **Lemma 5**, we get $x \in K$, which is a contradiction. Hence $x_u \notin K_S$. So that, $x_u + K_S \neq K_S$, which means that $x_u + K_S$ is a non zero element of the simple $R_S -$module $\frac{M_S}{K_S}$. Hence by using **Theorem E** and **Lemma 1**, we get $\frac{M_S}{K_S} = R_S(x_u + K_S) = R_S x_u + K_S =$ $(Rx)_S + K_S = (Rx + K)_S$. To show $\frac{M}{K} = Rx + K$. Let $m + K \in \frac{M}{N}$, for $m \in M$. Then $m_u + K_S \in \frac{M_S}{K_S} = (Rx)_S + K_S$. Hence, there exists $r \in R, w \in S$ such that $m_u + K_S = (rx)_w + K_S$ and so that $m_u - (rx)_w \in K_S$. Then $m_u - (rx)_w = l_d$, for some $l \in K, d \in S$. Hence $(wm - urx)_{uw} = l_d$. So that there exists $c \in S$ such that $cd(wm - urx) = cuwl$ or $cdwm = cdurx + cuwl$. As $c, d, w \in S$, they are non zero elements of $Z(R)$ and hence they are units of R, so $c^{-1}, d^{-1}, w^{-1} \in R$. Hence $m = w^{-1}urx + d^{-1}ul$ and thus $m + K = w^{-1}urx + d^{-1}ul + K \in Rx + K$, (since $w^{-1}ur \in R$ and $d^{-1}ul + K = K$). Hence $\frac{M}{K} \subseteq Rx + K$. But clearly, $Rx + K \subseteq \frac{M}{K}$, so $\frac{M}{K} = Rx + K = R(x + K)$. Hence $\frac{M}{K}$ is a simple $R -$module. Also, we see that $\frac{M_S}{K_S} =$

$(Rx)_S + K_S = (Rx + K)_S = (\frac{M}{K})_S$.

3: Using **Theorem F**, and the result in **(2)** we get, K is a maximal submodule of M if and only if $\frac{M}{K}$ is a simple $R -$module if and only if $\frac{M_S}{K_S}$ is a simple R_S – module if and only if K_S is a maximal submodule of M_S.

4: Let M be a simple R – module. To show M_S is a simple R_S – module. Let $0 \neq m_s \in M_S$, for $m \in M$ and $s \in S$. Then $m \neq 0$ (otherwise, $0 \neq m_s \in M_S$). Hence by **Theorem E**, we get $M = Rm$ and by using **Lemma 1** and **Lemma 2**, we get $M_S = R_S m_s$. Hence M_S is a simple R_S – module.

Conversely, let M_S be a simple R_S – module and $0 \neq x \in M$. Then for a fixed $t \in S$, we have $0 \neq x_t \in M_S$ (since if $x_t = 0$, then it can be shown that $x = 0$). As M_S is simple, we get $M_S = R_S x_t$. From **Lemma 2**, we get $M_S = (Rx)_S$ and since every element of $Z(R)$ is a unit, one can easily show that $M = Rx$. Hence M is a simple R – module.

5: Let $x_m \in J(M_S)$, where $x \in M, m \in S$. If K is any maximal submodule of M, then by the result of (**3**), K_S is a maximal submodule of M_S. Hence $x_m \in K_S$. So there exists $k \in K$ and $n \in S$ such that $x_m = k_n$ and then there exists $p \in S$ such that $pnx = pmk$. So from **Lemma 5**, we get $x \in K$ and thus $x \in J(M)$.Then $x_m \in (J(M))_S$ and hence $J(M_S) \subseteq (J(M))_S$. Now let $a_s \in (J(M))_S$, for $a \in J(M)$ and $s \in S$. If K^* is any maximal submodule of M_S, then by **Proposition 4**, we have $K^* = K_S$, for the submodule $K = \{x \in M : (sx)_s \in K^*\}$ of M and hence by the result of (**3**), we get K is a maximal submodule of M. Since $a \in J(M)$, so $a \in K$ and thus we get $a_s \in K_S = K^*$. Hence $a_s \in J(M_S)$, so that $(J(M))_S \subseteq J(M_S)$, which gives $J(M_S) = (J(M))_S$.

6: Using **Lemma 5** and by the above result of (**5**), we get $J(M) = 0$ if and only if $(J(M))_S = 0$ if and only if $J(M_S) = 0$.

7: Let K be cyclic submodule of M. To show K_S is cyclic submodule of M_S. So let $K = Rx$, for some $x \in M$. Then by **Lemma 1**, we get $K_S = (Rx)_S$. If $s \in S$ is an element of S, then $x_s \in M_S$.
From **Lemma 2**, we get $(Rx)_S = R_S x_s$.
Hence $K_S = R_S x_s$, which shows that K_S is cyclic submodule of M_S.

Conversely, let K_S be a cyclic submodule of M_S and $K_S = R_S a_m$, for some $a \in K$ and $m \in S$. From **Lemma 2**, we get $(Ra)_S = R_S a_m$. So that $K_S = (Ra)_S$. Then from **Lemma 5**, we get $K = Ra$. Hence K is a cyclic submodule of M □.

Now, it is the time to introduce the following definition:

Definition

We call M a centrally semiregular $R-$module if the localization M_S of M at every central multiplicative system S in R is a semiregular R_S-module.

Finally, we prove a sequence of theorems which determine the relations between regular, semiregular, centrally regular and centrally semiregular modules.

Theorem 9

If every non zero element of $Z(R)$ is a unit in R, then M is semiregular if and only if it is centrally semiregular.

Proof

Let M be semiregular. To show it is centrally semiregular. So let S be any central multiplicative system in R. To show M_S is semiregular. Let K^* be any cyclic submodule of M_S. Then by **Proposition 4**, we have $K^* = K_S$, for the submodule $K = \{a \in M : (sa)_S \in K^*\}$ of M and $s \in S$ is a fixed element. Then by the result of (**7**) of **Proposition 8**, we get K is a cyclic submodule of M and thus K lies above a direct summand of M. Hence there exists submodules P and Q of M such that $M = P \oplus Q$ with $P \subseteq K$ and $Q \cap K$ is a small submodule of Q. Then $P_S \subseteq K_S$ and by the result of (**1**) of **Proposition 8**, we get $(Q \cap K)_S$ is a small submodule of Q_S. Using the result in (**4**) of **Lemma 1**, we get $Q_S \cap K_S$ is a small submodule of Q_S. To show $M_S = P_S \oplus Q_S$. Clearly, $M = P + Q$ and by using the results in (**1**) and (**2**) of **Lemma 1**, we get $M_S = P_S + Q_S$. Next, to show $P_S \cap Q_S = 0$. Let $x' \in P_S \cap Q_S$, so there exists $p \in P, q \in Q$ and $m, n \in S$ such that $x' = p_m = q_n$. So, there exists $s \in S$ such that $snp = smq$ and hence by the result in (**1**) of **Lemma 5**, we get $p \in Q$. Hence $p \in P \cap Q = \{0\}$, that is, $p = 0$ and thus $x' = 0$, so that $P_S \cap Q_S = 0$. Hence $M_S = P_S \oplus Q_S$ and as $P_S \subseteq K_S$ and $Q_S \cap K_S$ is a small submodule of Q_S, we get that K_S lies above a direct summand of M_S, that is, K^* lies above a direct summand of M_S. Hence M_S is a semiregular R_S-module, that is, M is a locally semiregular $R-$module.

Conversely, let M be a centrally regular $R-$module. Then $S = Z(R) - \{0\}$ is a central multiplicative system in R and hence M_S is a semiregular R_S-module. To show M is a semiregular $R-$module. Let $K = Ra$ be any cyclic submodule of M. So by **Lemma 1** and **Lemma 2**, we get $K_S = (Ra)_S = R_S a_m$, for some $m \in S$. That is, K_S is a cyclic submodule of the semiregular R_S-module M_S. Hence K_S lies above a direct summand of M_S, that is, there exists submodules P^* and Q^* of M_S such that $M_S = P^* \oplus Q^*$ with $P^* \subseteq K_S$ and $Q^* \cap K_S$ is small in Q^*. But then by **Proposition 4**, there exists submodules P and Q of M such that $P^* = P_S$ and $Q^* = Q_S$, then we get $M_S = P_S \oplus Q_S$, $P_S \subseteq K_S$ and $Q_S \cap K_S$

is small in Q_S. Hence, $M_S = P_S + Q_S$, $P_S \cap Q_S = 0$, $P_S \subseteq K_S$ and $Q_S \cap K_S$ is small in Q_S, so that $M_S = (P+Q)_S$, $(P \cap Q)_S = 0$, $P_S \subseteq K_S$ and $(Q \cap K)_S$ is small in Q_S. Now using, **Lemma 5** and the result in (**1**) of **Proposition 8**, we get that $M = P + Q$, $P \cap Q = 0$, $P \subseteq K$ and $Q \cap K$ is small in Q, that is, $M = P \oplus Q$, $P \subseteq K$ and $Q \cap K$ is small in Q. Hence K lies above a direct summand of M and thus M is a semiregular $R-$module □ .

Combining **Theorem 9** with **Theorem B** and **Theorem C**, we get the following theorem:

Theorem 10

If every non zero element of $Z(R)$ is a unit and $J(M)$ is a small submodule of M, then the following conditions are equivalent:

1: M is semiregular.

2: M is centrally semiregular.

3: For every cyclic submodule N of M, there is an idempotent endomorphism f of M such that $f(M) \subseteq N$ and $(1-f)(N) \subseteq J(M)$.

4: For every finitely generated submodule N of M there exists an idempotent endomorphism f of M such that $f(M) \subseteq N$ and $(1-f)(N) \subseteq J(M)$.

5: Every finitely generated submodule of M lies above a direct summand of M.

6: For every cyclic submodule N of M, there is an idempotent endomorphism f of M such that $f(M) \subseteq N$ and $(1-f)(N)$ is a small submodule of M.

7: For every finitely generated submodule N of M there exists an idempotent endomorphism f of M such that such that $f(M) \subseteq N$ and $(1-f)(N)$ is a small submodule of M.

Combining **Theorem 7** with **Theorem D**, we get the following theorem:

Theorem 11:

If every non zero element of $Z(R)$ is a unit, then the following conditions are equivalent:

1: M is regular.

2: M is centrally regular.

3: Every finitely generated submodule of M is a direct summand of M.

4: For every finitely generated submodule N of M there exists an idempotent endomorphism f of M such that $f(M) = N$.

5: M is semiprimitive semiregular module.

6: Every submodule of M is a semiprimitive regular module.

References

1. Larsen M. D. and McCarthy P. J.: Multiplicative Theory of Ideals, **1971** Academic Press New York and London.
2. Jabbar A. K.: Centrally prime rings which arecommutative, *Journal of Kirk University*, **2006**, 1(2), 108-124.
3. Blyth T. S. : Module Theory, **1990,**Clarendon Press. Oxford.
4. Kamal M. A. and Yousef A. : On Principally Lifting Modules, *International Electronic Journal of Algebra*, **2007**, 2, 127-137.
5. Wang Y. and Sun Q. : A Note on $\oplus-$cofinitely supplemented modules, *International Journal of Mathematics and Mathematical Sciences*, **2007**, ID, 1-5.

6. Zeng Q. Y. and Shi M. H. : On closed weak supplemented modules, *Journal of Zhejiang University Science A*, **2006**, 7(2), 210-215.
7. Alkan M. and Ozcan A. C. : Semiregular modules with respect to a fully invariant submodules, Communications in Algebra, **2004**, 32, (11), 4285-4301.
8. Nebiyev C. and Pancar A. : Strongly $\oplus -$Supplemented Modules, *International Journal of Computational Cognition*, **2004**, 2(3), 57-61.
9. Tuganbaev A. A. : Rings of Quotients and Pierce Stalks, *Journal of Mathematical Sciences*, **2002**, 109 (3), 1569-1583.
10. Wisbauer R. : Foundations of Module and Ring Theory, Gordon and Breach Science Publishers, **1991**.

دەربارەی پێوەرە بەناوەندرێكەكان و پێوەرە بەناوەند نیمچەرێكەكان

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پوختە

لەم توێژینەوەیەدا پێناسەی دوو پێوەری نوێ پێشكەش كرا كە ناویانمان لێنا پێوەرە بەناوەندرێكەكان و پێوەرە بەناوەند نیمچەرێكەكان وچەندین خەسڵەتی ئەم دوو پێوەرەمان سەلماند. هەروەها چەندین مەرجمان پێشكەش كرد كە هەر یەكەیان پێوەرە رێكەكان (نیمچە رێكەكان) و پێوەرە بەناوەندرێكەكان (بەناوەند نیمچەرێكەكان) هاوسەنگ دەكات.

حول المقاسات المنتظمة مركزيا والمقاسات شبه المنتظمة مركزيا

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الخلاصة

فى هذا البحث قدمنا تعريفين لمقاسين جديدين سميناهما المقاسات المنتظمة مركزيا والمقاسات شبه المنتظمة مركزيا حيث تم برهنة العديد من خواص هذين المقاسين. كذلك تمكننا من الحصول على شروط عديدة و التي تجعل كل واحد منها من المقاسات المنتظمة (شبه المنتظمة) والمقاسات المنتظمة مركزيا (شبه المنتظمة مركزيا) مقاسات متكافئة.

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